eton press YOUR MATHEMATICAL SOLUTION and CASIO

SENIOR MATHEMATICS COMPETITION 2009

Final Round
Friday: 21st August 2009
Time allowed: 2 hours

INSTRUCTIONS

1. ***You may attempt all questions.***

2. ***Calculators may be used.***

3. ***Language dictionaries (non-electronic) are permitted, but no other reference material may be used.***

4. ***Diagrams are not necessarily drawn to scale.***

5. ***The mark allocated for each question is indicated. Marks will be awarded for clear reasoning; answers only will not necessarily earn full marks.***

Total: 50 marks

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NZAMT and CASIO

SENIOR MATHEMATICS COMPETITION 2009

1. For a real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.

   Eg $\lfloor 3{\cdot}6 \rfloor = 3$, $\lfloor -4{\cdot}1 \rfloor = -5$, $\lfloor 12 \rfloor = 12$ etc

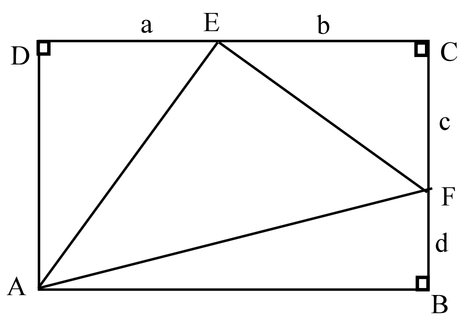
   Evaluate $\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \ldots + \lfloor \sqrt{2009} \rfloor$ [5]

2. The sum of the first six terms of a geometric sequence is 12, and the sum of the reciprocals of these six terms is 20.

   Find the product of these six terms. [5]

3. A circle with centre (1, 1) has a radius of 1 unit.

   If a point inside the circle is chosen at random, what is the probability that the y coordinate is more than twice the x coordinate? [5]

4. 

   ABCD is a rectangle.

   AEF is a triangle with lengths DE = a, EC = b, CF = c, and FB = d.

   Triangles ADE, ECF and FBA all have equal areas.

   (i) Show that $\frac{b}{a} = \frac{c}{d}$

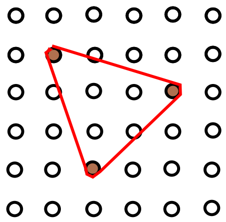
   and

   (ii) find the ratio $\frac{b}{a}$ [4]

5. How many right-angled triangles, with all sides an integer length in cm, have the shortest side of length 2009 cm? [4]

6. A 6 by 6 square lattice is formed by drilling holes in a wooden board.

Three pegs are placed at random in this lattice.



Find the probability that the three randomly chosen points of a 6 by 6 lattice will form a triangle. [5]

7. (i) Given $a^2 \geq b$, where a, b are natural numbers, prove that the necessary and sufficient condition for

$$\sqrt{a+\sqrt{b}} + \sqrt{a-\sqrt{b}}$$

to be a natural number is that there exists a natural number n such that

$$n^2 < a \leq 2n^2 \text{ and } b = 4n^2(a - n^2)$$

(ii) Find all values of natural numbers b such that

$$\sqrt{30+\sqrt{b}} + \sqrt{30-\sqrt{b}}$$

is a natural number. [7]

8. Find all non negative integers x, y, and z such that

$z^x = y^{2x}$

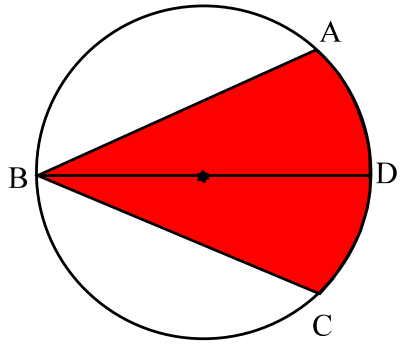
$2^z = 2 \times 4^x$

$x + y + z = 16$ [5]

9. A, B and C are three points on a circle, symmetrically placed about the diameter BD.

The shaded area ABCD is 50% of the total area.

What is the size of the angle ABC?



[5]

10. A function is defined recursively by $t_{n+2} = \frac{(t_n + t_{n+1})}{2}$; that is, each new term is the mean of the two previous two terms.

For example, when $t_1 = 1$ and $t_2 = 7$, we generate the following sequence

$$1, 7, 4, 5{\cdot}5, 4{\cdot}75, 5{\cdot}125, \ldots$$

Find the limit of the sequence for different starting values of t_1 and t_2.

[5]