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SENIOR MATHEMATICS COMPETITION 2011

Preliminary round Thursday
19th May 2011 Time
allowed 1 ½ hours

*Answers: These are a guide only. Markers should be generous in awarding
part marks for working which leads towards a good solution. Total: 50 marks*

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1. $F(x)$ will decrease most rapidly if x is a power of 2, and the next power of 2 greater than 2011 is 2048. Checking, if $x_1 = 2048$, then $x_n = 0$ for $n = 13$ which is less than 15. $\checkmark \checkmark$ [2]

2. 632135 must be the square of an integer with final digit 5; but any such square must end in "25". \checkmark
 512197, 276438 no square ends in 7 or 8. \checkmark
 636954 must be the square of an integer ending in 2 or 8; $(10n+2)^2 = 100n^2+40n+4$, so $4n$ must end in 5; and a similar contradiction for $(10n+8)$, $16n + 6$ must end in 5. $\checkmark \checkmark$ [4]

3. 2,3; $r = 1, 3, 5$; $r = 1$ $\checkmark \checkmark$ any one correct pair x \checkmark e.g. 11,13; $r = 9$ 23, 29; $r = 10$ [2]

4. LCM 2, 3, ...10 = 2520 \checkmark 2011000 = 40 (mod 252) \checkmark numbers are 20112120, 20114640, 20117160, 20119680 \checkmark [3]

5. By Pythagoras $AC = (\sqrt{5})x/2$ \checkmark $AY = (\sqrt{5})x/2 - x/2 = AX$; $AB \cdot BX = x(x - (\sqrt{5} - 1)x/2)$ \checkmark
 $= x^2(3 - (\sqrt{5})/2)$. $AX^2 = ((\sqrt{5} - 1)x/2)^2$ \checkmark $= x^2(6 - 2(\sqrt{5})/4 = x^2(3 - (\sqrt{5})/2 = AB \cdot BX$ \checkmark [4]

6. $\text{Det} \geq 0$ so $(4(a - 2))^2 - 4(4)(-8a^2 + 14a + 31) \geq 0$, \checkmark $a^2 - 2a - 3 \geq 0$, $(a - 3)(a + 1) \geq 0$, $a \geq 3$ or $a \leq -1$. \checkmark
 Let r and s be the roots, then $r + s = -4(a - 2)/4$ and $rs = (-8a^2 + 14a + 31)/4$, \checkmark $r^2 + s^2 = (r + s)^2 - 2rs$
 $= (10a^2 - 22a - 23)/2$ \checkmark $= (10(a - 11/10)^2 - 23 - 121/10)/2$, which is a minimum when $a = 11/10$ where roots are not real. \checkmark Quadratic is symmetrical, $a = 11/10$ is closer to $a = 3$ than to $a = -1$, so $a = 3$. \checkmark [6]

7. $OA = OB = OC = r$, radius \perp tangent; $\angle AOB = 2\angle ACB$, \angle at centre = $2\angle$ at circumf. same arc; $\checkmark \checkmark$
 $\angle AOF = \angle C$, $\tan C = \tan(\angle AOF) = AF/r$; so $AF = r \tan C$. \checkmark $\Delta = (r^2 \tan C)/2$, \checkmark quadrilateral area = $r^2 \tan C$,
 so area of triangle DEF is $r^2(\tan A + \tan B + \tan C)$. As required. \checkmark [5]

8. Without loss of generality let $AC = BC = 1$, \checkmark so $AB = r$, $AD = r/2$. By cosine rule $\cos A = \cos B$
 $= (1^2 + r^2 - 1^2)/(2 \times 1 \times r)$ \checkmark $= r^2/2r = r/2$, \checkmark and $\cos C = (1^2 + 1^2 - r^2)/(2 \times 1 \times 1) = (2 - r^2)/2 = 1 - r^2/2$.
 So $\cos A + \cos B + \cos C = 1 + r - r^2/2$. As required. \checkmark [4]

9. $P(M') = 3/6$, $P(T') = 4/6$ so $P(D \text{ on toss } 3) = 3/6 \times 4/6 \times 1/6$; \checkmark on toss 6: $P(D) = (3/6)^2(4/6)^2(5/6)(1/6)$;
 on toss $3n$: $P(D) = (3/6)^n(4/6)^n(5/6)^{n-1}(1/6)$. \checkmark So $P(D) = \sum(1 + 5/18 + (5/18)^2 + \dots + (5/18)^n + \dots)/18$ \checkmark
 $= (1/(1 - (5/18)))/18 = 1/13$ \checkmark [4]

10. (a) $\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A = 2\sin A \cos A \cos A + (\cos^2 A - \sin^2 A)\sin A =$
 $3\sin A(1 - \sin^2 A) - \sin^3 A = 3\sin A - 4\sin^3 A$. As required. $\checkmark \checkmark$ [2]
 (b) $\sin 54^\circ = \cos 36^\circ$, let $x = \sin 18^\circ$, \checkmark then $3x - 4x^3 = 1 - 2x^2$ \checkmark so $(x - 1)(4x^2 + 2x - 1) = 0$;
 $x \neq 1$ so $x = (-2 \pm 2\sqrt{5})/8 = (-1 \pm \sqrt{5})/4$. As required. \checkmark [3]
 (c) This is ratio $a:b$ where Δ sides $a:1:1$ has angles $36^\circ, 72^\circ, 72^\circ$ and Δ sides $b:1:1$ has angles $108^\circ, 36^\circ, 36^\circ$
 By sine or cosine rule $a^2/b^2 = 4\sin^2 18^\circ/4\sin^2 54^\circ$, \checkmark so $a/b = \sin 18^\circ/\sin 54^\circ = \sin 18^\circ/(3\sin 18^\circ - 4\sin^3 18^\circ)$
 $= 1/(3 - 4\sin^2 18^\circ)$ \checkmark $= 2/(3 + \sqrt{5}) = (3 - \sqrt{5})/2$. As required \checkmark [3]

11. If n is between 90 and 100 \checkmark $f(n) = f(f(n + 1)) = f(n + 1)$, \checkmark $f(90) = f(91) = \dots = f(101) = 91$, \checkmark
 if $n < 90$ then any n will be moved to a corresponding number 11 greater in the next interval, \checkmark
 so that $f(10) = 91$. \checkmark [5]

12. Diameter of semicircle = $\sqrt{2}$ x diameter of outside arc since isosceles Δ . \checkmark So small circle has half area of large circle. \checkmark So outer semicircle has same area as inner quarter circle. Subtracting the central area from each gives area of lune = area of triangle. As required. \checkmark [3]

