eton press YOUR MATHEMATICAL SOLUTION and CASIO

SENIOR MATHEMATICS COMPETITION 2009

SOLUTIONS

Final Round
Friday: 21st August 2009
Time allowed: 2 hours

INSTRUCTIONS

1. ***You may attempt all questions.***
2. ***Calculators may be used.***
3. ***Language dictionaries (non-electronic) are permitted, but no other reference material may be used.***
4. ***Diagrams are not necessarily drawn to scale.***
5. ***The mark allocated for each question is indicated. Marks will be awarded for clear reasoning; answers only will not necessarily earn full marks.***

Total: 50 marks

Proudly supported by eton press YOUR MATHEMATICAL SOLUTION

NZAMT and CASIO

1. For a real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.

Eg $\lfloor 3{\cdot}6 \rfloor = 3$, $\lfloor -4{\cdot}1 \rfloor = -5$, $\lfloor 12 \rfloor = 12$ etc

Evaluate $\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \ldots + \lfloor \sqrt{2009} \rfloor$ [5]

Let n be a natural number, $\lfloor \sqrt{n^2} \rfloor = n$

Also $\lfloor \sqrt{n^2+1} \rfloor = \lfloor \sqrt{n+2} \rfloor = \ldots = \lfloor \sqrt{(n+1)^2 - 1} \rfloor = n$

This is because $\lfloor \sqrt{x} \rfloor$ takes only integer values and $\lfloor \sqrt{n^2} \rfloor = n$ and $\lfloor \sqrt{(n^2+1)} \rfloor = n+1$

so if we group the numbers

$$\underbrace{(\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor)}_{n=1,\ 3\text{ terms}} + \underbrace{(\lfloor \sqrt{4} \rfloor + \ldots + \lfloor \sqrt{8} \rfloor)}_{n=2,\ 4\text{ terms}} + \underbrace{(\lfloor \sqrt{9} \rfloor + \ldots}_{n=3,\ 7\text{ terms}}$$ and so on

In each group there are $(n+1)^2 - n^2$ terms each with an integer value n.

So the required sum is Note *($44^2 = 1936$)*

$$\sum_{n=1}^{43} [(n+1)^2 - n^2]n + 44 \times (2009 - 1936 + 1)$$

$$= \sum_{n=1}^{43} [(n+1) + n][(n+1) - n]n + 3256$$

$$= \sum_{n=1}^{43} [2n+1]n + 3256$$

$$= 2\sum_{n=1}^{43} [n^2] + \sum_{n=1}^{43} [n] + 3256$$

$$= 2\left(\frac{43 \times 44 \times 87}{6}\right) + \frac{43 \times 44}{2} + 3256$$

$$= 59070$$

Answer by Chang Joo Park

2. The sum of the first six terms of a geometric sequence is 12, and the sum of the reciprocals of these six terms is 20.

Find the product of these six terms. [5]

Starting term = a , common ratio = r

First six terms in geometric sequence are $a, ar, ar^2, ar^3, ar^4, ar^5$

the sum of the first six terms

$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 = 12 \qquad ①$$

the sum of the reciprocals of these six

$$\frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \frac{1}{ar^2} + \frac{1}{ar^4} + \frac{1}{ar^5} = 20 \qquad ②$$

looking at eqn 2

$$\frac{r^5}{ar^5} + \frac{r^4}{ar^5} + \frac{r^3}{ar^5} + \frac{r^2}{ar^5} + \frac{r}{ar^5} + \frac{1}{ar^5} = 20$$

$$\frac{r^5 + r^4 + r^3 + r^2 + r + 1}{ar^5} = 20$$

looking at ①

$$a\,(1 + r + r^2 + r^3 + r^4 + r^5) = 12$$

Let $1 + r + r^2 + r^3 + r^4 + r^5 = \mathrm{k}$

then ② tells us $\mathrm{k} = 20ar^5$ *and* ① tells us $k = \frac{12}{a}$

$$20ar^5 = \frac{12}{a}$$

$$a^2r^5 = \frac{12}{20}$$

we want to find $a \times ar \times ar^2 \times ar^3 \times ar^4 \times ar^5 = a^6r^{15}$

$$= (a^2r^5)^3$$

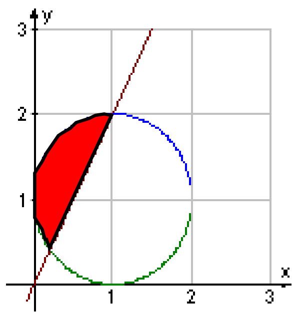
$$= \left(\frac{3}{5}\right)^3$$

$$= \frac{27}{125} \quad or \ 0.216$$

So- Young Park

3. A circle with centre (1, 1) has a radius of 1 unit.

If a point inside the circle is chosen at random, what is the probability that the *y* coordinate is more than twice the *x* coordinate? [5]



Picking $y > 2x$

Drawing $y = 2x$ line.

$\therefore$ Probability of inside shaded.

Circle formula is $(x-1)^2 + (y-1)^2 = 1$

Intercept i (1,2) = A

ii $y = 2x$

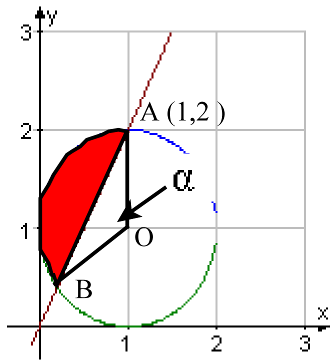
$(x-1)^2 + (2x-1)^2 = 1$

$x^2 - 2x + 1 + 4x^2 - 4x + 1 = 1$

$5x^2 - 6x + 1 = 0$

$(5x - 1)(x - 1) = 0$

$x = 1$ or $x = \frac{1}{5}$ $\therefore$ B is at $(\frac{1}{5}, \frac{2}{5})$



The length of AB $= \sqrt{\left(1-\frac{1}{5}\right)^2 + \left(2-\frac{2}{5}\right)^2}$

$= \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{8}{5}\right)^2}$

$= \sqrt{\left(\frac{80}{5}\right)}$

$= \frac{4\sqrt{5}}{5}$

By cosine rule

$AB^2 = 1^2 + 1^2 - 2\cos\alpha$

$\frac{80}{25} = 2 - 2\cos\alpha$

$\frac{-3}{5} = \cos\alpha$

$\alpha = 126 \cdot 9^o$

Area shaded $= \pi \times \frac{\alpha}{360} - \frac{1}{2}\sin\alpha$

$= 1 \cdot 11 - \frac{1}{2} \times \frac{4}{5}$

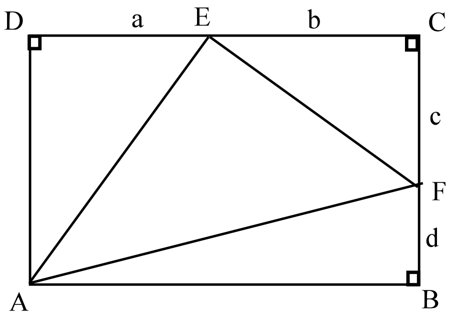
$= 1 \cdot 11 - 0 \cdot 4$

$= 0 \cdot 707$

Probability required $= \frac{0 \cdot 707}{\pi}$

$= 0 \cdot 225$

Song-Kyeong Lee

4. 

ABCD is a rectangle.

AEF is a triangle with lengths DE = a, EC = b, CF = c, and FB = d.

Triangles ADE, ECF and FBA all have equal areas.

(i) Show that $\frac{b}{a} = \frac{c}{d}$

and

(ii) find the ratio $\frac{b}{a}$ [4]

i Given $|\Delta ADE| = |\Delta ABF|$

gives ½ a(c + d) = ½ d(a + b)

$\Leftrightarrow \quad ac + ad = ad + bd$

$\Leftrightarrow \quad ac = bd$

$\Leftrightarrow \quad \frac{b}{a} = \frac{c}{d}$

ii $|\Delta ADE| = |\Delta ECF|$

gives ½ a(c + d) = ½ bc

$$c + d = \frac{bc}{a}$$

$$\frac{c}{c} + \frac{d}{c} = \frac{b}{a}$$

$$1 + \frac{d}{c} = \frac{b}{a}$$

$$now \frac{c}{d} = \frac{b}{a}$$

so

$$1 + \frac{a}{b} = \frac{b}{a}$$

Let $\frac{b}{a} = x$ $\quad Then,\ x = 1 + \frac{1}{x}$

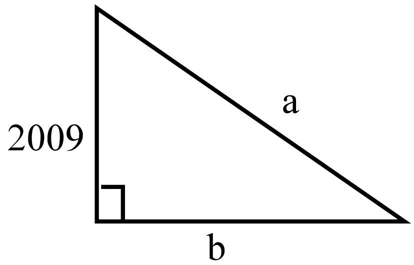
$$x^2 - x + 1 = 0$$

Positive root is $x = \frac{1 + \sqrt{5}}{2}$

$$so\ \frac{b}{a} = \frac{1 + \sqrt{5}}{2}$$

5. How many right-angled triangles, with all sides an integer length in cm, have the shortest side of length 2009 cm?

Let longest side = a and other short side = b [4]



Pythagoras gives $a^2 - b^2 = 2009^2$

$(a + b)(a - b) = (7^2 \times 41)^2$ So

a + b and a - b must both be odd as the product of two odd numbers is odd

consider table

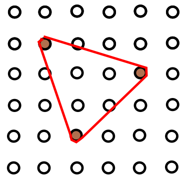
a + b	a - b	a	b	outcome
$7^4.41^2$	1	2 018 041	2 018 040	yes
$7^3.41^2$	7	288 295	288 288	yes
$7^2.41^2$	7^2	41 209	41 160	yes
7.41^2	7^3	6 055	5712	yes
$7^4.41$	41	49 241	49 200	yes
$7^3.41$	7. 41	7 175	6 888	yes
7^4	41^2	2 041	360	no as b < 2 009

this completes the possibilities as a + b > a - b, hence **six** triangles are possible.

6. A 6 by 6 square lattice is formed by drilling holes in a wooden board.

Three pegs are placed at random in this lattice.

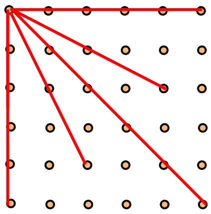
Find the probability that the three randomly chosen points of a 6 by 6 lattice will form a triangle.



[5]

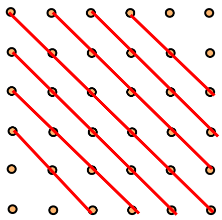
3 points will always form a triangle unless they are in a straight line.

The number of ways 3pegs can be placed in a lattice of 36 holes is $^{36}C_3 = 7140$



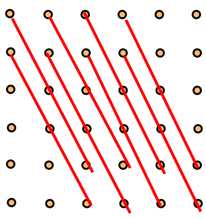
3 pegs in a horizontal line is $6 \times {}^{6}C_3$

3 pegs in a vertical line is $6 \times {}^{6}C_3$



Diagonals

$2\,(\,2\,(\,{}^{3}C_3 + {}^{4}C_3 + {}^{5}C_3\,) + {}^{6}C_3\,) = 100$



Other diagonals that can occur, 8x4 different orientations, that gives 32 lines.

Number of ways to make a line

$6 \times {}^{6}C_3 + 6 \times {}^{6}C_3 + 100 + 32 = 372$

$$\text{probability of a triangle} = 1 - \frac{372}{^{36}C_3}$$

$$= 0.947899$$

Daniel Redmond

7. (i) Given $a^2 \geq b$, where *a, b* are natural numbers, prove that the necessary and sufficient condition for

$$\sqrt{a+\sqrt{b}} + \sqrt{a-\sqrt{b}}$$

to be a natural number is that there exists a natural number *n* such that

$$n^2 < a \leq 2n^2 \text{ and } b = 4n^2(a-n^2)$$

(ii) Find all values of natural numbers *b* such that

$$\sqrt{30+\sqrt{b}} + \sqrt{30-\sqrt{b}}$$

is a natural number. [7]

Let m = $\sqrt{a+\sqrt{b}} + \sqrt{a-\sqrt{b}}$ is a natural number,

where $a^2 \geq b$, and *a* and *b* are natural numbers.

Squaring gives $m^2 = 2a + 2\sqrt{a^2-b}$ (1)

$\Leftrightarrow m^2 - 2a = 2\sqrt{a^2-b}$ (2)

$\Leftrightarrow (m^2 - 2a)^2 = 4(a^2-b)$ (3)

$\Leftrightarrow m^4 = 4(am^2-b)$ (4)

in (4), *RHS is even, then* $m = 2n$, *n is a natural number*

then (4) $4n^2 = 4an^2 - b$

$\Leftrightarrow b = 4n^2(a-n^2)$ (5)

(5) and $b \geq 1 \Rightarrow a - n^2 > 0 \Rightarrow n^2 < a$ *

from (3) $\Rightarrow (2n)^2 - 2a \Rightarrow 0$

$\Rightarrow a \leq 2n^2$ *

combining the *'s gives $n^2 < a \leq 2n^2$
and (5) $b = 4n^2(a - n^2)$ } (6)

<u>Converslely</u> if *a, b, n* satisfy (6)

Let $m = 2n \Rightarrow$ establish (4) and hence (3). Since $m^2 - 2a \geq 0$, then (2) follows, and

hence $m = \sqrt{a+\sqrt{b}} + \sqrt{a-\sqrt{b}}$

(2) from $a^2 \geq b$, $n^2 < a \leq 2n^2$, $b = 4n^2(a - n^2)$

$\Rightarrow \sqrt{30+\sqrt{b}} + \sqrt{30-\sqrt{b}}$ to be a natural number

$a^2 = 900 \geq b$

$n^2 < 30 \leq 2n^2$

$\Rightarrow n = 4 \text{ or } 5$

$b^2 = 4n^2(30 - n^2)$ if $n = 4$, $b = 896$ if $n = 5$, $b = 500$

Marker

8. Find all non negative integers x, y, and z such that

$z^x = y^{2x}$ ①

$2^z = 2\times 4^x$ ②

$x + y + z = 16$ ③ [5]

Find first, x, y, z $\in \mathbb{N}$ where x, y, z $\neq 0$

① $\Leftrightarrow z = y^2$

② $\Leftrightarrow 2^z = 2^{2x+1}$

$\therefore 2x + 1 = z$

$x = ½ (y^2 - 1)$

$x + y + z = 16$

$½ (y^2 - 1) + y + y^2 = 16$

$3y^2 + 2y - 33 = 0$

$(3y + 11)(y - 3) = 0$

$\therefore y = 3 \because y \in \mathbb{N}$

giving z = 9 and x = 4 (x , y , z) = (4, 3, 9)

If x = 0 ② : $2^z = 2 \because z = 1$, y = 16 - 0-1 =15

(x , y , z) = (0, 15, 1)

If y = 0 $\Leftrightarrow$ z = 0 from ① then x =16 from ③

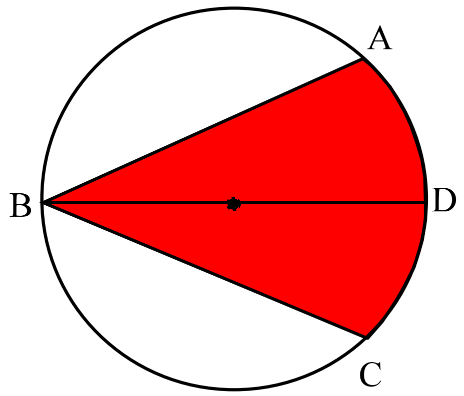
but then $2^z = 1 \neq 2\times 4^{16}$ Hence z, y $\neq 0$

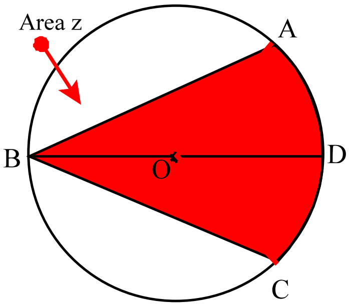
David Shin

9. A, B and C are three points on a circle, symmetrically placed about the diameter BD.

The shaded area ABCD is 50% of the total area.

What is the size of the angle ABC?





Without loss of generality let radius of circle be 1 unit.

Area of circle = π, Area BAD = $^1/_4\ \pi$ by symmetry and ABCD occupying 50% of the total area.

Area of segment z = $\frac{1}{2}\pi - {}^1/_4\ \pi = {}^1/_4\ \pi$

Let $\angle ABD = \theta$, and $\angle AOB = x$, then $\angle AOD = 2\theta = \angle ABC$ by symmetry

and $2\theta + x = \pi$ radians

Area segment = $\frac{1}{2} r^2 (\theta - \sin\theta)$

$\therefore$ Area z gives $\frac{1}{2}(x - \sin x) = {}^1/_4\ \pi$.

$\therefore \quad x - \sin x = \frac{1}{2}\pi$

Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

f(x) = $x - \sin x - \frac{1}{2}\pi$, $f'(x) = 1 - \cos x$

$$x_{n+1} = x_n - \frac{x_n - \sin x_n - \frac{\pi}{2}}{1 - \cos x_n}$$

Put $x_0 = 1$ we get $x_1 = 4.0722$

repeating we get to x = 2.30988

Thus $\angle AOD = 2\theta = \pi - 2.30988$

thus $\theta = 0.8317$radians

Matthew Ogden

10. A function is defined recursively by $t_{n+2} = \frac{(t_n + t_{n+1})}{2}$; that is, each new term is the mean of the two previous two terms.

For example, when $t_1 = 1$ and $t_2 = 7$, we generate the following sequence

$$1, 7, 4, 5{\cdot}5, 4{\cdot}75, 5{\cdot}125, \ldots$$

Find the limit of the sequence for different starting values of t_1 and t_2. [5]

$2\, t_{n+2} = t_n + t_{n+1}$

$2\, t_{n+2} + x\, t_{n+1} = (1 + x)\, t_{n+1} + t_n$

let x be such that

$$\frac{2}{x} = 1 + x$$

$$\Leftrightarrow x^2 + x - 2 = 0$$

$$x = -2 \text{ or } x = 1$$

$\therefore\ 2\,(t_{n+2} - t_{n+1}) = -(t_{n+1} - t_n)$

let $T_n = 2\,(t_{n+2} - t_{n+1})$, *then* $T_{n+1} = -\frac{1}{2} T_n$

$\therefore T_n = T_1 (-\frac{1}{2})^n = (t_2 - t_1)(-\frac{1}{2})^n$

$\therefore t_n = (t_n - t_{n-1}) + (t_{n-1} - t_{n-2}) + \ldots + (t_2 - t_1) + t_1$

$$= \sum_{i=1}^{n-1} T_n + t_1$$

$$\lim_{n\to\infty} t_n = \frac{(t_2 - t_1)}{1 - (-\frac{1}{2})} + t_1$$

$$= \frac{t_1 + 2\,t_3}{3}$$

Sicong Zhang